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Normal subobjects of topological groups and of topological semi-Abelian algebras

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Abstract

As any category $Gp(\mathbb{E})$ of internal groups in a given category $\mathbb{E}$, the category $Gp(Top)$ of topological groups possesses the strong algebraic property of *protomodularity* which carries intrinsic notions of normal subobject and of centrality. Here we explicit and investigate these intrinsic notions in the category $Gp(Top)$. We extend these results to any category $Top^{\mathbb{T}}$ of topological semi-Abelian algebras.

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0. Introduction

Besides the fact that the forgetful functor $U : Gp(Top) \to Gp$ has very good properties, the category $Gp(Top)$ of (not necessarily Hausdorff) topological groups inherits many algebraic aspects of the category Gp of groups on the account of sharing with it the property of being *protomodular*, a conceptual context [8] within which there is, among other things, an intrinsic notion of normal subobject and of centrality. Of course, in this context, any

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kernel map is a normal subobject, but in general there are far more normal subobjects than kernel maps. Since, moreover, $Gp(Top)$ is *regular* [2], there is an internal notion of exact sequence, and consequently [9] (but more surprisingly) all the classical homological lemmas (short five lemma, Noether isomorphisms, 3×3 lemma, or even snake lemma) do hold in it. In that way, although the category $Gp(Top)$ contains some objects absolutely without topological interest (as the indiscrete topological groups for instance), it appears to have much better global properties than its (more usually considered) subcategory of Hausdorff topological groups. Moreover, the presence of these uninteresting objects allows us to measure or to characterize some very interesting properties inside $Gp(Top)$ (see Corollary 1.2, Remark 1.2 and Proposition 4.4 for instance).

Beyond the topological characterization of the normal subobjects (Proposition 3.1), we shall focus our attention on three points dealing with this protomodularity condition:

(1) Given (G, T_G) and (K, T_K) two topological groups, the topological product is not only the weakest topology with respect to the continuity of the two canonical homomorphic projections:

$$G \leftarrow G \times K \rightarrow K$$

but also the finest topology with respect to the continuity of the two canonical homomorphic injections:

$$G \rightarrowtail G \times K \leftarrowtail K.$$

(2) Given (G, T_G) a non-Abelian topological group, and I a normal subgroup of G, we shall list, at least in the particular case of clopen topological groups, all the topologies T_I on I, such that

$$(I, T_I,) \rightarrowtail (G, T_G)$$

becomes a normal subobject in $Gp(Top)$. More precisely, there are as many topologies as normal subgroups J of G satisfying: $[I, \overline{\{1_G\}}] \subset J \subset I \cap \overline{\{1_G\}}$.

(3) Any pointed regular and finitely cocomplete protomodular category $\mathbb{C}$ being given, it is possible to develop an intrinsic commutator theory, see [10,12,11]. When moreover $\mathbb{C}$ is exact, we have always, for any pair of objects (X, Y) of $\mathbb{C}$ the classical inclusion $[X, Y] \subset X \cap Y$ (again [10,11]). Here we shall exhibit the category $Gp(Top)$ as a counterexample to this inclusion in the regular (but non-exact) context. This was the initial aim of this work.

A last section shows that all the results of this article, except the specific characterization of normal subobjects, extend from topological groups to topological semi-Abelian algebras in the sense of [4], among the main instances of which there are the topological rings.

1. The category *Gp*(*Top*)

Let $Gp(Top)$ be the category of (not necessarily Hausdorff) topological groups. We shall gather here the main categorical facts. First, the forgetful functor $U : Gp(Top) \rightarrow Gp$ is topological, see [17,1,4], and consequently cotopological. It therefore has all the good lifting properties one can hope for. This implies in particular that $Gp(Top)$ is finitely complete and cocomplete, and that the functor U has a left adjoint right inverse $D : Gp \to Gp(Top)$ which associates with any group G the topological group (G, T_G^0) where T_G^0 is the discrete topology and also a right adjoint right inverse $F : Gp \to Gp(Top)$ which associates with any group G the topological group (G, T_G^1) where T_G^1 is the indiscrete topology. Therefore the functor U is a fibration, or more precisely the pair (U, F) is a fibered reflection (see [6] for instance), a morphism $f : (G, T_G) \to (G, T'_{G'})$ being *Cartesian* when the following square is a pullback:

$$\begin{array}{ccc} (G, T_G) & \xrightarrow{f} & (G', T_{G'}) \\ \downarrow & & \downarrow \\ (G, T_G^1) & \xrightarrow[f]{} & (G', T_{G'}^1) \end{array}$$

This means that the only open sets of T_G are the inverse images by f of the open sets of $T_{G'}$. When this is the case, we shall denote by T_G^f this topology on G. It is clear that when $f = i$ is the inclusion of a subgroup, T_G^i is precisely the induced topology. The class of Cartesian maps is stable by composition, by pullback and contains the homeomorphic isomorphisms. If the map $g = f.h$ and the map f are Cartesian, then the map h is Cartesian as well. Any map $f : (G, T_G) \to (G', T_{G'})$ in $Gp(Top)$ has a canonical decomposition $f = f_c.f_i$ with f_c Cartesian and f_i U-invertible (i.e. such that $U(f_i)$ is an isomorphism). Moreover, any commutative square whose one pair of parallel arrows is Cartesian and whose image by U is a pullback is itself a pullback. Let us first emphasize the following more specific result:

Proposition 1.1. *Given any topological group (G, T_G), the closure $\overline{\{1_G\}}$ of the unit element 1_G is such that its induced topology is indiscrete. Accordingly the functor $F : Gp \to Gp(Top)$ admits the functor $C : Gp(Top) \to Gp$ as a right adjoint, where $C(G, T_G) = \overline{\{1_G\}}$.*

Proof. It is well known that the closure $\overline{\{1_G\}}$ is a normal subgroup of G. On the other hand, any non-empty closed set W of $\overline{\{1_G\}}$ is closed in T_G. So if $1_G \in W$, then $W = \overline{\{1_G\}}$. If not, there is an $x \in W \subset \overline{\{1_G\}}$ such that $x^{-1}W$ is closed in $\overline{\{1_G\}}$ and contains 1_G, so that $x^{-1}W = \overline{\{1_G\}}$, and $W = x.\overline{\{1_G\}} = \overline{\{1_G\}}$. Accordingly, $\overline{\{1_G\}}$ has no other non-empty closed set but itself, and the induced topology is consequently indiscrete. So that we have $(\overline{\{1_G\}}, T^i) = F(\overline{\{1_G\}})$. Now take a group L and a continuous homomorphism $l : (L, T_L^1) \to (G, T_G)$. Then $l^{-1}(\overline{\{1_G\}})$ is a non-empty closed set of T_L, so that $l^{-1}(\overline{\{1_G\}}) = L$, and h has a factorization $\bar{h} : L \to \overline{\{1_G\}}$. $\square$

So, we have here a remarkable sequence of three adjunctions: $D \dashv U \dashv F \dashv C$.

On the other hand, the regular epimorphisms in the category $Gp(Top)$ are just the surjective open homomorphisms. The category $Gp(Top)$ is regular since the regular epis are stable by pullback, and any effective equivalence relation in $Gp(Top)$ (i.e. any kernel pair) has a coequalizer [2]. Given any regular epimorphism $h : (G, T_G) \to (H, T_H)$ and any subobject (I, T_I) of (G, T_G), the *direct image* of (I, T_I) by h is the pair $(h(I), T_{h(I)}^q)$ where

$h(I)$ is the ordinary image subgroup of $h(I)$ in H, and $T^q_{h(I)}$ is the relative quotient topology, namely such that V is an open set in $T^q_{h(I)}$ if and only if $h^{-1}(V)$ is an open set in T_I. Finally, it is clear that a Cartesian surjective homomorphism is necessarily a regular epimorphism.

Lemma 1.1. *If the map $g = f.h$ is Cartesian and the map h is a Cartesian regular epimorphism, then the map f is Cartesian as well.*

Proof. Consider the canonical decomposition $f = f_c.f_i$. Since $g = f.h = f_c.f_i.h$ is Cartesian, then the map $f_i.h$ is Cartesian. Since f_i is U-invertible, f_i is a monomorphism, and, on the other hand, h and $f_i.h$ are Cartesian above the same surjective homomorphism. Accordingly $f_i.h$ is a regular epimorphism, and so is f_i. Being a monomorphism and a regular epimorphism, the map f_i is an isomorphism, and the map f is Cartesian. □

1.1. Gp(Top) is protomodular

The category *Gp*(*Top*) is also protomodular [7] as any category *Gp*($\mathbb{E}$) with $\mathbb{E}$ finitely complete. Recall that a finitely complete pointed category $\mathbb{C}$ is *protomodular*, provided that, given any split epimorphism f, the pair (*ker* f, s) in the following pullback is jointly strongly epic, see [8,9]:

$$\begin{tikzcd} K[f] \arrow[r, "ker\, f"] \arrow[d] & X \arrow[d, shift left, "f"] \\ 1 \arrow[r, "\alpha_Y"'] & Y \arrow[u, shift left, "s"] \end{tikzcd}$$

This means that, when we are given any monomorphism $j : I \rightarrowtail X$, if the pullbacks of j along *ker* f and s are both isomorphisms, then the map j is itself an isomorphism. In other words, this means that, in $\mathbb{C}$, the subobject 1_X is the supremum of the pair (*ker* f, s) of subobjects of X. One of the first striking consequence of the protomodularity of *Gp*(*Top*) is the following:

Proposition 1.2. *Given any continuous homomorphism $f : (X, T_X) \to (Y, T_Y)$, split by a continuous homomorphism s in Gp(Top), a group homomorphism $h : X \to H$ is continuous from (X, T_X) to (H, T_H), if and only if the group homomorphisms $h.\,ker\, f : (K[f], T^i) \to (H, T_H)$ and $h.s : (Y, T_Y) \to (H, T_H)$ are continuous*:

$$\begin{tikzcd} (K[f], T^i) \arrow[r, "ker\, f"] \arrow[d] & (X, T_X) \arrow[r, dotted, "h"] \arrow[d, shift left, "f"] & (H, T_H) \\ 1 \arrow[r, "\tau_Y"'] & (Y, T_Y) \arrow[u, shift left, "s"] & \end{tikzcd}$$

Proof. Let us consider the finest topology T on X which makes (X, T) a topological group and the maps *ker* $f : (K[f], T^i) \to (X, T)$ and $s : (Y, T_Y) \to (X, T)$ continuous. This is the topology which makes universally continuous any group homomorphism $h : X \to H$ such that the group homomorphisms $h.\mathit{ker}\, f : (K[f], T^i) \to (H, T_H)$ and $h.s : (Y, T_Y) \to (H, T_H)$ are continuous. In particular the map $Id_X : (X, T) \rightarrowtail (X, T_X)$ is continuous. It is a monomorphism in *Gp*(*Top*), and its pullbacks along *ker* f and s are both isomorphisms, as the commutativity of the following diagram shows immediately:

$$\begin{tikzcd}
(K[f], T^i) \arrow[r, tail, "\mathit{ker}\, f"] \arrow[d, "Id_{(K[f],T^i)}"'] & (X, T) \arrow[d, tail, "Id_X"] & (Y, T_Y) \arrow[l, tail, "s"'] \arrow[d, "Id_{(Y,T_Y)}"] \\
(K[f], T^i) \arrow[r, tail, "\mathit{ker}\, f"'] & (X, T_X) & (Y, T_Y) \arrow[l, tail, "s"]
\end{tikzcd}$$

Consequently Id_X is an homeomorphism, and $T = T_X$. □

We have in particular:

Corollary 1.1. *Given* (G, T_G) *and* (K, T_K) *two topological groups, a group homeomorphism* $h : G \times K \to H$ *is continuous from the topological product* $(G, T_G) \times (K, T_K)$ *to* (H, T_H) *if and only if the composite* $h.r_G$ *and* $h.l_K$ *are continuous, where* $l_G : G \rightarrowtail G \times K$ *and* $r_K : K \rightarrowtail G \times K$ *are the canonical continuous injections.*

1.2. Gp(*Top*) *is homological*

As a pointed, regular and protomodular category, *Gp*(*Top*) is *homological*, and all the homological lemmas do hold inside it, see [9] and also [3]. An exact sequence is then a sequence:

$$\begin{tikzcd}
1 \arrow[r] & (K, T_K) \arrow[r, tail, "k"] & (G, T_G) \arrow[r, two heads, "h"] & (H, T_H) \arrow[r] & 1
\end{tikzcd}$$

where h is a surjective open homomorphism and k is the (necessarily Cartesian) kernel map of h. In other words $T_K = T_K^k$ and $T_H = T_H^q$.

As a first homological aspect of the protomodularity of *Gp*(*Top*), let us mention:

Proposition 1.3. *Consider the following morphism of exact sequence. Then the map* f *is Cartesian if and only if the induced map* ϕ *is Cartesian*:

$$\begin{tikzcd}
1 \arrow[r] & (K, T_K^k) \arrow[r, tail, "k"] \arrow[d, "\phi"'] & (G, T_G) \arrow[r, two heads, "h"] \arrow[d, "f"'] & (H, T_H) \arrow[r] \arrow[d, "Id_H"] & 1 \\
1 \arrow[r] & (K', T_{K'}^{k'}) \arrow[r, tail, "k'"'] & (G', T_{G'}) \arrow[r, two heads, "h'"'] & (H, T_H) \arrow[r] & 1
\end{tikzcd}$$

Proof. The left-hand square is always a pullback, so that if f is Cartesian, the map ϕ is also Cartesian. Conversely, let us consider the following diagram induced by the canonical decomposition $f = f_c.f_i$:

$$\begin{tikzcd}
1 \arrow[r] & (K, T_K^k) \arrow[r, tail, "k"] \arrow[d, "Id_K"'] & (G, T_G) \arrow[r, two heads, "h"] \arrow[d, "f_i"'] & (H, T_H) \arrow[r] \arrow[d, "Id_H"] & 1 \\
1 \arrow[r, dotted] & (K, T_K^k) \arrow[r, tail, "\bar{k}"'] \arrow[d, "\phi"'] & (G, T_G^f) \arrow[r, two heads, "\bar{h}"'] \arrow[d, "f_c"'] & (H, T_H) \arrow[r, dotted] \arrow[d, "Id_H"] & 1 \\
1 \arrow[r] & (K', T_{K'}^{k'}) \arrow[r, tail, "k'"'] & (G', T_{G'}) \arrow[r, two heads, "h'"'] & (H, T_H) \arrow[r] & 1
\end{tikzcd}$$

The map $\bar{h}$ is a regular epi, since $\bar{h}.f_i = h$ is a regular epi. On the other hand, since the map f_i is U-invertible, the image by U of the lower left-hand square is a pullback in *Gp*. Since the pair (ϕ, f_c) is a pair of Cartesian maps, this same square is itself a pullback in *Gp*(*Top*). Accordingly the middle row is exact. By the short five lemma, the map f_i is an isomorphism in *Gp*(*Top*), and consequently an homeomorphism. Thus $T_G = T_G^f$, and f is Cartesian. □

From this, we can derive a characterization of the Cartesian regular epimorphisms:

Corollary 1.2. *A regular epimorphism* $h : (G, T_G) \twoheadrightarrow (G', T_{G'})$ *is Cartesian if and only if the induced topology on Kerh is indiscrete.*

Proof. The terminal map $\tau_H : (H, T_H) \to 1$ is Cartesian if and only if the topology T_H is indiscrete. Now consider the following diagram and apply the previous proposition:

$$\begin{tikzcd}
1 \arrow[r] & (Ker\, h, T^i) \arrow[r, tail, "i"] \arrow[d] & (G, T_G) \arrow[r, two heads, "h"] \arrow[d, "h"'] & (G', T_{G'}) \arrow[r] \arrow[d, "Id"] & 1 \\
1 \arrow[r] & 1 \arrow[r, tail] & (G', T_{G'}) \arrow[r, two heads, "Id"'] & (G', T_{G'}) \arrow[r] & 1
\end{tikzcd} \qquad \square$$

Remark 1.1. According to Proposition 1.1, in the following exact sequence, the map h is Cartesian:

$$\begin{tikzcd}
1 \arrow[r] & (\overline{\{1_G\}}, T^i) \arrow[r, tail, "i"] & (G, T_G) \arrow[r, two heads, "h"] & (G/\overline{\{1_G\}}, T^q) \arrow[r] & 1
\end{tikzcd}$$

Remark 1.2. This property of the map h transforms the trite observation on the (Hausdorff, see [5,14]) topological group $(G/\overline{\{1_G\}}, T^q)$ that any non-empty open or closed set is the union of its elements into the non-trivial fact that any non-empty open or closed set in (G, T_G) is necessarily an arbitrary union of cosets of $\overline{\{1_G\}}$.

In the same order of ideas we have:

Proposition 1.4. *Suppose that we are given the following pullback with h' a regular epi, then f is Cartesian if and only ψ is Cartesian*:

$$\begin{tikzcd}
(X,T_X) \arrow[r, "h"] \arrow[d, "f"'] & (Y,T_Y) \arrow[d, "\psi"] \\
(X',T_{X'}) \arrow[r, two heads, "h'"'] & (Y',T_{Y'})
\end{tikzcd}$$

Proof. Only the "only if" part needs proof. Consider the following diagram with $\psi = \psi_c.\psi_i$ the canonical decomposition:

$$\begin{tikzcd}
(X,T_X) \arrow[r, two heads, "h"] \arrow[d, "f"'] \arrow[drr, two heads] & (Y,T_Y) \arrow[d, "\psi"'] \arrow[dr, "\psi_i"] & \\
 & & (\bar{Y},T_{\bar{Y}}) \arrow[dl, "\psi_c"] \\
(X',T_{X'}) \arrow[r, two heads, "h'"'] & (Y',T_{Y'}) &
\end{tikzcd}$$

Then, ψ_i being U-invertible, the image by U of the lower quadrangle is a pullback in *Gp*. Accordingly, since the parallel arrows ψ_c and f are Cartesian, the quadrangle in question is itself a pullback. The map $\psi_i.h$ is then a regular epi, and thus a strong epi, with ψ_i (as a U-invertible map) a monomorphism. Thus ψ_i is an isomorphism in $Gp(Top)$. □

Finally we have:

Theorem 1.1. *Consider, in $Gp(Top)$, the following morphism of exact sequences*:

$$\begin{tikzcd}
1 \arrow[r] & (K,T_K^k) \arrow[r, tail, "k"] \arrow[d, "\phi"'] & (G,T_G) \arrow[r, two heads, "h"] \arrow[d, "f"'] & (H,T_H) \arrow[r] \arrow[d, "\psi"] & 1 \\
1 \arrow[r] & (K',T_{K'}^{k'}) \arrow[r, tail, "k'"'] & (G',T_{G'}) \arrow[r, two heads, "h'"'] & (H',T_{H'}) \arrow[r] & 1
\end{tikzcd}$$

(1) *when ϕ and ψ are Cartesian, then f is Cartesian too,*
(2) *when f and ψ are Cartesian, then ϕ is Cartesian too,*
(3) *when f and ϕ are Cartesian, with moreover ϕ a regular epi, then ψ is Cartesian too.*

Proof. Consider the following decomposition, where the lower right-hand square is a pullback:

$$\begin{tikzcd}
1 \arrow[r] & (K,T_K^k) \arrow[r, tail, "k"] \arrow[d, "\phi"'] & (G,T_G) \arrow[r, two heads, "h"] \arrow[d, "f_1"'] & (H,T_H) \arrow[r] \arrow[d, "Id_H"] & 1 \\
1 \arrow[r, dotted] & (K',T_{K'}^{k'}) \arrow[r, tail, "\bar{k}'"'] \arrow[d, "Id_{K'}"'] & (P,T_P) \arrow[r, two heads, "\bar{h}'"'] \arrow[d, "f_2"'] & (H,T_H) \arrow[r, dotted] \arrow[d, "\psi"] & 1 \\
1 \arrow[r] & (K',T_{K'}^{k'}) \arrow[r, tail, "k'"'] & (G',T_{G'}) \arrow[r, two heads, "h'"'] & (H',T_{H'}) \arrow[r] & 1
\end{tikzcd}$$

On the one hand, the map $\bar{h}'$ is a regular epimorphism, since $\bar{h}'.f_1 = h$ is itself a regular epi. On the other hand there is a unique map $\bar{k}'$ which completes the middle row as an exact sequence and makes the upper left-hand square a pullback.

(1) If ϕ is Cartesian, then according to Proposition 1.3, the map f_1 is Cartesian. Since ψ is Cartesian, the map f_2 is Cartesian as well, the Cartesian maps being stable by pullback. Consequently $f = f_2.f_1$ is Cartesian.
(2) If ψ is Cartesian, then the map f_2 is Cartesian as well. But since $f = f_2.f_1$ is Cartesian, the map f_1 is Cartesian. Thus ϕ is Cartesian by Proposition 1.3.
(3) If ϕ is a Cartesian regular epi, then f_1 is Cartesian, and a regular epi by Proposition 8 in [9]. Since $f = f_2.f_1$ is Cartesian, the map f_2 is Cartesian. Then, according to the previous proposition, the map ψ is Cartesian. □

2. Normal subobjects

Let us recall the following definition, see [8]:

Definition 2.1. Given any finitely complete category $\mathbb{C}$, a map $m : I \to X$ is normal to an equivalence relation R on X when $m^{-1}(R)$ is the indiscrete relation ∇_I on I (i.e. the kernel equivalence relation of the terminal map $\tau_I : I \to 1$) and when the following induced map $(I, \nabla_I) \to (X, R)$ in the category *Rel*($\mathbb{C}$) of equivalence relations in $\mathbb{C}$ is fibrant:

$$\begin{tikzcd} I \times I \arrow[r, "\bar{m}"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & R \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ I \arrow[r, "m"'] & X \end{tikzcd}$$

This means that any of the commutative squares in the previous diagram is a pullback. The fact that a map m is normal implies that m is necessarily a monomorphism. This definition is an intrinsic way to express that I is an equivalence class of R. Normal subobjects are stable by intersection, pullback and product. *When moreover the category $\mathbb{C}$ is protomodular, the map m is normal to at most one equivalence relation*, and consequently the fact to be normal, in this kind of category, becomes a property. *When the category $\mathbb{C}$ is pointed protomodular, the normal subobjects of an object X are in bijection with the internal equivalence relations on X* [8]. When $\mathbb{C}$ is homological, the normal subobjects are stable by direct image [3].

Given now any finitely complete category $\mathbb{E}$, the category $Gp(\mathbb{E})$ of internal groups in $\mathbb{E}$ is clearly pointed (i.e. with a zero object). It was shown to be protomodular in [8], and consequently it yields a natural notion of normal subobject. We shall present here an internal group as an object X of $\mathbb{E}$ endowed with a division, i.e. a binary operation $d : X \times X \to X$ with a left unit $e : 1 \to X$ internally satisfying: $d(e, x) = x$, $d(x, x) = e$ and $d(d(x, y), d(x, z)) = d(y, z)$. We shall need specifically the map $d_* : X \times X \to X$, internally corresponding to $d_*(y,x) = d(d(y,x),x) = x^{-1}.y.x$. Let (I, d_I) be any subgroup of (X, d) in $Gp(\mathbb{E})$, and let us consider now the following pullback:

$$\begin{tikzcd} \Gamma_X^I \arrow[r, "\delta"] \arrow[d, "j"'] & I \arrow[d, "i"] \\ X \times X \arrow[r, "d"'] & X \end{tikzcd}$$

This is clearly internally corresponding to $\Gamma_X^I = \{(u,v) \in X \times X \mid u^{-1}.v \in I\}$.

Lemma 2.1. *The map $j : \Gamma_X^I \rightarrowtail X \times X$ defines, on the object X in $\mathbb{E}$, an internal equivalence relation to which the subobject $i : I \rightarrowtail X$ is normal in $\mathbb{E}$.*

Proof. Straightforward, thanks to the Yoneda embedding. □

Of course, there is no reason why Γ_X^I would be a subgroup of the group $X \times X$. Categorically speaking, this is the case if and only if, in the category $\mathbb{E}$, the left-hand side vertical map in the following pullback is an isomorphism, where the map $d_{X\times X}$ denotes the division of the group product:

$$\begin{tikzcd} P \arrow[rr] \arrow[d] & & \Gamma_X^I \arrow[d, "j"] \\ \Gamma_X^I \times \Gamma_X^I \arrow[r, "j\times j"'] & (X \times X)^2 \arrow[r, "d_{X\times X}"'] & X \times X \end{tikzcd}$$

Proposition 2.1. *The subobject $i : I \rightarrowtail X$ is normal in $Gp(\mathbb{E})$ if and only if Γ_X^I is a subgroup of the group $X \times X$.*

Proof. When Γ_X^I is a subgroup of the group $X \times X$, then $i : I \rightarrowtail X$ becomes normal to Γ_X^I in $Gp(\mathbb{E})$. Conversely suppose $i : I \rightarrowtail X$ is normal to some equivalence relation R in $Gp(\mathbb{E})$. Then, thanks to the Yoneda embedding, we can check that $R \simeq R \cap \Gamma_X^I \simeq \Gamma_X^I$. □

There is also the following characterization, which is the internal translation, thanks to the Yoneda embedding, of a well known result in the set theoretical context:

Proposition 2.2. *The map $i : I \rightarrowtail X$ is normal in $Gp(\mathbb{E})$ if and only if, in the category $\mathbb{E}$ the left-hand side vertical map in the following pullback is an isomorphism*:

$$\begin{tikzcd} J \arrow[rr] \arrow[d] & & I \arrow[d, "i"] \\ I \times X \arrow[r, "i\times X"'] & X \times X \arrow[r, "d_*"'] & X \end{tikzcd}$$

Proof. This is the categorical translation of the fact that, for all $(y,x) \in I \times X$, the element $x^{-1}.y.x$ must be in I. □

We know [8] that, in a protomodular category, an object X is Abelian if and only if the diagonal $s_0 : X \rightarrowtail X \times X$ is normal. We have also:

Corollary 2.1. *An internal group X in $\mathbb{E}$ is Abelian if and only if $d_* = p_0 : X \times X \to X$.*

3. Normal subobjects in the category of topological groups

Let us come back to the category $Gp(Top)$. Let (X, T_X) a topological group, and (I, T_I) a subobject in $Gp(Top)$. Then the topology T_I is such that the inclusion $i : I \rightarrowtail X$ is continuous, but of course T_I is not necessarily the topology T_I^i induced by T_X on I. According to the previous characterization, we obtain:

Proposition 3.1. *A subobject $i : (I, T_I) \rightarrowtail (X, T_X)$ is normal in the category $Gp(Top)$ if and only if:*

(1) *the subgroup I is a normal subgroup of X,*
(2) *the map $d_* : I \times X \to I$; $(y, x) \mapsto x^{-1}.y.x$ is continuous from the topological product $(I, T_I) \times (X, T_X)$ to (I, T_I).*

The second condition shows that there is no reason why the morphism $Id : (X, T_X^0) \to (X, T_X)$ would be a normal subobject in general. Of course, any normal subgroup I of X produces a normal subobject $i : (I, T_I^i) \rightarrowtail (X, T_X)$ by means of the induced topology. But any normal subobject is not necessarily of this form. For instance, as soon as I is a central subgroup, we have $d_* = p_I : I \times X \to I$ since $x^{-1}.y.x = y$; accordingly, in this case, any subobject of the form $i : (I, T_I) \rightarrowtail (X, T_X)$ is normal. So that, *when A is an Abelian group, and I is any subgroup, any continuous inclusion $i : (I, T_I) \rightarrowtail (A, T_A)$ produces a normal subobject*. This is the case in particular for $Id_A : (A, T_A^0) \to (A, T_A)$. This example is particularly interesting since it emphasizes, provided T_A is not discrete, that there are, in $Gp(Top)$, normal subobjects which are not kernels. We have the following obvious, but useful characterization:

Proposition 3.2. *A monomorphism $i : (I, T_I) \rightarrowtail (X, T_X)$ in $Gp(Top)$ is a kernel if and only if it is normal and Cartesian.*

We have also the following remarkable observation:

Proposition 3.3. *Consider a normal (to the equivalence relation (R, T_R)) subobject $i : (I, T_I) \rightarrowtail (G, T_G)$ in $Gp(Top)$. Then $\overline{\{1_I\}}$ is a normal subgroup of $\overline{\{1_G\}}$, and the closure $\overline{\{1_R\}}$ with respect to T_R is equal to $\{(u, v) \in \overline{\{1_G\}}^2 \mid u^{-1}.v \in \overline{\{1_I\}}\}$.*

Proof. It is a straightforward consequence of the fact that the functor $C : Gp(Top) \to Gp$, having a left adjoint, is left exact, and consequently preserves the pullbacks and the normal subobjects. □

4. The normal subobjects of clopen topological groups

Given a non-Abelian topological group (G, T_G) and any normal subgroup I of G, it would certainly be interesting to measure to what extend, besides the induced topology, it is possible to endow I with a topology T_I in such a way that the inclusion $i:(I, T_I) \rightarrowtail (G, T_G)$ produces a normal subobject in *Gp*(*Top*). We shall achieve this in a very particular and simple situation which will prepare the construction of the counterexample given in the next section:

Definition 4.1. A topological group (G, T_G) is said to be clopen when any closed subset is open.

More generally, a topological space (X, T) will be said clopen when any closed subset is open. The topology of such a space is always produced by a partition of the set X. Indeed, the only connected clopen topological spaces are the indiscrete ones; on the other hand, the connected component of any point x of a clopen topological space (X, T) being closed, it is also open, and consequently the connected component of x is the smaller open and closed set containing x. As a consequence the full subcategory Top_{co} of *Top* whose objects are the clopen topological spaces is isomorphic to the category *Rel* whose objects are the pairs (X, R) of a set and an equivalence relation, and whose maps are the applications which preserve the relations. Clearly the subcategory Top_{co} is stable (inside *Top*) under limits and Cartesian maps. Accordingly the full subcategory Γ_{co} of *Gp*(*Top*) whose objects are the clopen topological groups is nothing but $Gp(Top_{co})$ and is consequently protomodular, and stable (inside *Gp*(*Top*)) under limits and Cartesian maps (i.e. when $f:(G, T_G) \to (G', T_{G'})$ is Cartesian and $(G', T_{G'})$ clopen, then (G, T_G) is clopen). Moreover since $\Gamma_{co} = Gp(Top_{co}) = Gp(Rel) = Rel(Gp)$ (i.e. the category of internal equivalence relations in *Gp*), it is regular. So the category Γ_{co} is homological. On the other hand, any discrete topological group (G, T_G^0) and any indiscrete topological group (G, T_G^1) is in Γ_{co}. Straightforward also is the following characterization:

Proposition 4.1. *The following conditions are equivalent*:

(1) *the topological group* (G, T_G) *is clopen,*
(2) *the closure* $\overline{\{1_G\}}$ *open,*
(3) *the quotient topology on* $G/\overline{\{1_G\}}$ *is discrete.*

Proof. Clearly the condition (1) implies the condition (2). Let us suppose (2). Since the quotient map h is open, the set $h(\overline{\{1_G\}}) = \{1\}$ is open, and the quotient topology is discrete. Let us suppose (3). We know, by Remark 1.1, that the map $h:(G, T_G) \twoheadrightarrow (G/\overline{\{1_G\}}, T^q)$ is always Cartesian. If the quotient topology is discrete, then the quotient is clopen. Now the clopen topological groups are stable under Cartesian maps, and (G, T_G) is itself clopen. $\square$

A topological group (G, T_G) is Hausdorff if and only if the closure $\overline{\{1_G\}} = \{1_G\}$. Consequently the only Hausdorff clopen topological groups are the discrete ones. The clopen topological groups are thus given by a pair (G, K) of a group G and a normal subgroup K, the open sets of the clopen topology Θ_G^K being given by the arbitrary unions of cosets of K. In that way, the topology Θ_G^K appears as the finest topology on G such that $\overline{\{1_G\}} = K$. Actually we have more:

Proposition 4.2. *The inclusion functor $I : \Gamma_{co} \to Gp(Top)$ has a right adjoint $Co : Gp(Top) \to \Gamma_{co}$, i.e. is a full coreflective embedding.*

Proof. Let (G, T_G) be a topological group. Then Remark 1.2 precisely means that the following map is continuous:

$$\big(G, \Theta_G^{\overline{\{1_G\}}}\big) \xrightarrow{Id} (G, T_G)$$

This map is the required universal arrow. Indeed, let $h : (H, \Theta_H^K) \to (G, T_G)$ be a continuous homomorphism. Then $h^{-1}(\overline{\{1_G\}})$ is a closed set in H containing the unit element. Accordingly we have $K \subset h^{-1}(\overline{\{1_G\}})$, so that we have a continuous homomorphism $\bar{h} : (H, \Theta_H^K) \to (G, \Theta_G^{\overline{\{1_G\}}})$. $\square$

We shall prove now a first result concerning the normal subobjects in Γ_{co}.

Proposition 4.3. *Let be given a normal subobject $i : (I, T_I) \rightarrowtail (G, T_G)$ in $Gp(Top)$. If (I, T_I) and (G, T_G) are in Γ_{co}, then i is normal in Γ_{co}.*

Proof. Let (R, T_R) the equivalence relation in $Gp(Top)$ to which i is normal. We must check that T_R is in Γ_{co}. Since the functor $Co : Gp(Top) \to \Gamma_{co}$, having a left adjoint, is left exact, the map i is also normal to the image $(\bar{R}, T_{\bar{R}})$ of (R, T_R) by Co. But $Gp(Top)$ is protomodular, so that $(R, T_R) \simeq (\bar{R}, T_{\bar{R}})$. $\square$

We have now an easy characterization of the normal subobjects in Γ_{co}.

Proposition 4.4. *A subobject $i : (I, \Theta_I^J) \rightarrowtail (G, \Theta_G^K)$ in Γ_{co}, is normal if and only if J is a normal subgroup of G, such that $[I, K] \subset J \subset I \cap K$.*

Proof. We shall work more easily in $Rel(Gp)$. So let $i : (I, J) \rightarrowtail (G, K)$ be a subobject normal to the equivalence relation (R, S) on (G, K). Since the functors U and C are left exact, then we have necessarily $R = \Gamma_G^I$ and $S = \Gamma_K^J$. And consequently Γ_K^J is a normal subgroup of Γ_G^I. This means that we have: $(x, y)^{-1}.(a, b).(x, y) \in \Gamma_K^J$ for any $(a, b) \in \Gamma_K^J$ and any $(x, y) \in \Gamma_G^I$. In other words, this means that $x^{-1}.a^{-1}.x.y^{-1}.b.y \in J$, provided that $a \in K$, $b \in K$, $a^{-1}.b \in J$ and $x^{-1}.y \in I$. In particular, taking $x = y \in G$, $a = 1$ and $b \in J$, we obtain $x^{-1}.b.x \in J$ which means that J is a normal subgroup of G. Taking $a = b \in K$, $x = 1$ and $y \in I$, we obtain $b^{-1}.y^{-1}.b.y \in J$ which means $[I, K] \subset J$.

Conversely suppose that J is a normal subgroup of G and $[I, K] \subset J \subset I \cap K$. We are going to show that $i : (I, J) \rightarrowtail (G, K)$ is normal to (Γ_G^I, Γ_K^J). Clearly $i : I \rightarrowtail G$ is normal to Γ_G^I, and $j : J \rightarrowtail K$ is normal to Γ_K^J. It remains to show that $(\Gamma_G^I, \Gamma_K^J) \in \Gamma_{co}$, i.e. that Γ_K^J is a normal subgroup of Γ_G^I. But we have:

$$z = x^{-1}.a^{-1}.x.y^{-1}.b.y = \big(x^{-1}.a^{-1}.b.x\big).\big(x^{-1}.b^{-1}.x.y^{-1}.b.y.x^{-1}.x\big).$$

Now J is normal in G and $a^{-1}.b \in J$, so that $x^{-1}.a^{-1}.b.x \in J$. On the other hand $b^{-1}.(x.y^{-1}).b.(y.x^{-1}) \in [I, K]$ since $b \in K$ and $y.x^{-1} \in I$. But we have $[I, K] \subset J$, whence $b^{-1}.(x.y^{-1}).b.(y.x^{-1}) \in J$ and $(x^{-1}.b^{-1}.x.y^{-1}.b.y.x^{-1}.x) \in J$. Consequently z is certainly in J. $\square$

So, given a clopen topological group (G, Θ_G^K) and any normal subgroup I of G, there are as many clopen topological normal subobjects of (G, Θ_G^K) above I as normal subgroups J of G such that $[I, K] \subset J \subset I \cap K$. The associated normal subobject with this J is the continuous inclusion $i : (I, \Theta_I^J) \rightarrowtail (G, \Theta_G^K)$.

5. Commutator theory in *Gp*(*Top*)

Any protomodular category $\mathbb{C}$ is Mal'cev [8], and as soon as it is finitely complete and regular, it admits an intrinsic commutator theory [10]. When moreover $\mathbb{C}$ is exact, then, for any pair of equivalence relations (R, S) on an object Z of $\mathbb{C}$, the following classical inclusion $[R, S] \subset R \cap S$ holds (again [10,11]). We shall now exhibit the category *Gp*(*Top*) as a counterexample to this inclusion in the regular (but non-exact) context.

The category *Gp*(*Top*) is not only protomodular, but also strongly protomodular, so that the commutator theory for the topological congruences reduces to the one for the normal subobjects [11]. Let us recall that two subobjects $i : X \rightarrowtail G$ and $j : Y \rightarrowtail G$ commutes in a pointed protomodular category $\mathbb{C}$ when there exists a (necessarily unique) map $\varphi : X \times Y \to G$ making the following diagram commute:

$$\begin{tikzcd}
& X \arrow[dl, "l_X"'] \arrow[dr, "i"] & \\
X \times Y \arrow[rr, "\varphi"'] & & G \\
& Y \arrow[ul, "r_Y"] \arrow[ur, "j"'] &
\end{tikzcd}$$

Corollary 1.1 anticipated the fact that there were no peculiar topological informations with respect to the commutation of subobjects in *Gp*(*Top*). Indeed, let a topological group (G, T_G) and two normal subobjects $i : (X, T_X) \rightarrowtail (G, T_G)$, $j : (Y, T_Y) \rightarrowtail (G, T_G)$ be given.

Proposition 5.1. *The pair* (i, j) *commutes in Gp*(*Top*) (*see* [10]) *if and only if the pair* (i, j) *commutes in Gp, i.e. if and only if we have* $x.y = y.x$ *for any pair* $(x, y) \in X \times Y$.

Proof. Thanks to the Corollary 1.1, it is straightforward that, if the pair (i, j) commutes in *Gp*, the induced group homomorphism $\theta : X \times Y \to G$; $(x, y) \rightarrowtail x.y$ is necessarily continuous from the topological product $(X, T_X) \times (Y, T_Y)$ to (G, T_G). $\square$

Let us translate, however, the general construction of the commutator. Let a topological group (G, T_G) and two normal subobjects (X, T_X), (Y, T_Y) be given. First, consider the following diagram, where $(Q[X,Y],T)$ is the colimit of the diagram made of the plain arrows, and where l_X and r_Y denote the (continuous) canonical inclusion in the product:

$$\begin{tikzcd}
 & (X,T_X) \arrow[dl, "l_X"'] \arrow[d, dotted, "\bar{\phi}_X"] \arrow[dr, "i"] & \\
(X,T_X)\times(Y,T_Y) \arrow[r, dotted, "\bar{\phi}"'] & (Q[X,Y],T) & (G,T_G) \arrow[l, dotted, "\bar{\psi}"] \\
 & (Y,T_Y) \arrow[ul, "r_Y"] \arrow[u, dotted, "\bar{\phi}_Y"'] \arrow[ur, "j"'] &
\end{tikzcd}$$

Clearly the two maps $\bar{\phi}_X$ and $\bar{\phi}_Y$ are completely determined by the pair $(\bar{\phi},\bar{\psi})$. Moreover the map $\bar{\phi}$ induces, for the pair $((\bar{\psi}(X),T^q_{\bar{\psi}(X)}),(\bar{\psi}(Y),T^q_{\bar{\psi}(Y)}))$ of the direct images by $\bar{\psi}$, a continuous commutation. On the other hand, the map $\bar{\psi}$ is a regular epimorphism which measures the lack of commutation of the pair (i,j). More precisely we have following result, [10]:

Proposition 5.2. *The map $\bar{\psi}$ is the universal regular epimorphism in the category $\mathbb{C}=Gp(Top)$ which makes the images of the pair (i,j) continuously commute. The map $\bar{\psi}$ is an isomorphism if and only if the pair (i,j) commutes continuously.*

Secondly, set the following definition:

Definition 5.1. The topological commutator of the pair (i,j) is the kernel of the map $\bar{\psi}$ in $Gp(Top)$ and is denoted by $[(X,T_X),(Y,T_Y)]$.

The lack of topological specification with respect to the commutation of subobjects in $Gp(Top)$ (see Proposition 5.1) is reflected by the following observation: since the functor U is both left and right exact, we have certainly:

$$\big[(X,T_X),(Y,T_Y)\big]=\big([X,Y],T^i\big)$$

where T^i is the topology induced by T_G on $[X,Y]$. All the major properties of the classical commutator are satisfied [10], in particular:

(1) the commutativity,
(2) the monotony in each variable,
(3) for any regular epimorphism h, we have:

$$h\big(\big[(X,T_X),(Y,T_Y)\big]\leqslant\big[\big(h(X),T^q_{h(X)}\big),\big(h(Y),T^q_{h(Y)}\big)\big]\big).$$

However, in $Gp(Top)$, we shall have no longer something analogous with $[X,Y]\leqslant X\cap Y$ which, in general, requires the fact that the homological category $\mathbb{C}$ in question is not only regular, but also Barr exact [10]:

Proposition 5.3. *In the homological category $Gp(Top)$ the commutator of two normal subobjects is not embedded in their intersection.*

Proof. Consider a group G and its centre Z. Since $[G, Z] = \{1_G\}$, by 4.5, we have a normal subobject $Id : (G, T_G^0) = (G, \Theta_G^{\{1_G\}}) \rightarrowtail (G, \Theta_G^Z)$ in $Gp(Top)$. Then: $(G, T_G^0) \cap (G, T_G^0) = (G, T_G^0)$, while: $[(G, T_G^0), (G, T_G^0)] = ([G, G], T^i) = ([G, G], \Theta_{[G,G]}^{Z\cap[G,G]})$.

So, if $Z \cap [G, G]$ is not equal to $\{1_G\}$, the inclusion $i : [G, G] \rightarrowtail G$ is clearly not continuous from the non-discrete topological group $([G, G], \Theta_{[G,G]}^{Z\cap[G,G]})$ to the discrete one (G, T_G^0).

Now take the orthogonal group $\mathcal{O}_2(\mathbb{R})$, then its centre Z is reduced to $\{Id, -Id\}$, while the commutator $[\mathcal{O}_2(\mathbb{R}), \mathcal{O}_2(\mathbb{R})]$ is $\mathcal{O}_2^+(\mathbb{R})$. Then the group $G = \mathcal{O}_2(\mathbb{R})$ realizes the previous situation, since we have $Z \cap [\mathcal{O}_2(\mathbb{R}), \mathcal{O}_2(\mathbb{R})] = Z \neq \{Id\}$. □

6. The case of topological semi-Abelian algebras

The semi-Abelian categories are the finitely cocomplete pointed exact protomodular categories, see [16]. The pointed protomodular varieties are necessarily semi-Abelian, and were characterized in [13]:

Proposition 6.1. *A variety $\mathcal{V}$ is semi-Abelian if and only if its theory has a unique constant* 1*, binary terms $t_1, \ldots, t_n$, and $(n+1)$-ary term t satisfying the identities $t(x, t_1(x, y), \ldots, t_n(x, y)) = y$ and $t_i(x, x) = 1$ for each $i = 1, \ldots, n$.*

Of course the variety Rg of rings or any variety $\mathcal{V}$ of Ω-groups [15] is semi-Abelian. The notion of topological semi-Abelian algebra has been introduced in [4] as a topological model of a given semi-Abelian theory $\mathbb{T}$ (= model of the theory $\mathbb{T}$ provided with a topology which makes all the operations of the theory continuous). It is then natural to investigate whether the previous results on topological groups extend to topological semi-Abelian algebras, with first of all the category $Rg(Top)$ of topological rings in mind. The answer is positive except for the characterization of normal subobjects.

6.1. General setting

Let $Top^{\mathbb{T}}$ be the category of topological algebras of a semi-Abelian theory $\mathbb{T}$. Then again the category $Top^{\mathbb{T}}$ is finitely complete and cocomplete, pointed, regular and protomodular [4]. Let $U : Top^{\mathbb{T}} \to Set^{\mathbb{T}}$ denote the forgetful functor. Then, on the model of $Gp(Top)$ this functor is topological [4]. Consequently it has a left adjoint right inverse $D : Set^{\mathbb{T}} \to Top^{\mathbb{T}}$ which associates with any algebra A the topological algebra (A, T_A^0) where T_A^0 is the discrete topology. It has also a right adjoint right inverse $F : Set^{\mathbb{T}} \to Top^{\mathbb{T}}$ which associates with any algebra A the topological algebra (A, T_A^1) where T_A^1 is the indiscrete topology. It is a fibration, and this determines in $Top^{\mathbb{T}}$ a notion of Cartesian map which satisfies exactly the same properties as the Cartesian maps in $Gp(Top)$. We have also an analogue of Proposition 1.1:

Proposition 6.2. *Given any topological algebra (A, T_A), the closure $\overline{\{1_A\}}$ of the unit element 1_A is a normal subalgebra such that its induced topology is indiscrete. Accordingly* *the functor $F: Set^{\mathbb{T}} \to Top^{\mathbb{T}}$ admits the functor $C: Top^{\mathbb{T}} \to Set^{\mathbb{T}}$ as a right adjoint, where $C(A, T_A) = \overline{\{1_A\}}$.*

Proof. The fact that the closure $\overline{\{1_A\}}$ is a normal subalgebra of A is shown in [4]. On the other hand, any non-empty closed set W of $\overline{\{1_A\}}$ is closed in T_A. If $1_A \in W$, then $W = \overline{\{1_A\}}$. If not, there is an $a \in W \subset \overline{\{1_A\}}$. Consider now the following map: $\varphi : \overline{\{1_A\}} \to \overline{\{1_A\}}$ defined by:

$$\varphi_a(x) = t\big(1, t_1(x, a), \dots, t_n(x, a)\big).$$

We have then: $\varphi_a(1) = a$ and $\varphi_a(a) = t(1, t_1(a, a), \dots, t_n(a, a)) = t(1, 1, \dots, 1) = 1$. Moreover the map $\varphi_a : (\overline{\{1_A\}}, T^i) \to (\overline{\{1_A\}}, T^i)$ is continuous. Then $\varphi_a^{-1}(W)$ is a closed set of $\overline{\{1_G\}}$ which contains 1_A. Accordingly it is equal to $\overline{\{1_G\}}$. Thus $a \in \varphi_a^{-1}(W)$, and $\varphi_a(a) = 1_A \in W$. So that $W = \overline{\{1_A\}}$. Consequently, $\overline{\{1_A\}}$ has no other non-empty closed set but itself, and the induced topology is consequently indiscrete. In these conditions, we have $(\overline{\{1_A\}}, T^i) = F(\overline{\{1_A\}})$.

Now take an algebra L and a continuous homomorphism $l : (L, T_L^1) \to (A, T_A)$. Then $l^{-1}(\overline{\{1_A\}})$ is a non-empty closed set of T_L, so that $l^{-1}(\overline{\{1_A\}}) = L$, and h has a factorization $\bar{h} : L \to \overline{\{1_A\}}$. □

Since $Set^{\mathbb{T}}$ is pointed and protomodular, there is a bijection between the normal subalgebras of A and the congruences on A. Let us denote by R_A^{Hf} the congruence associated with $\overline{\{1_A\}}$, we shall need it below.

The functor $U : Top^{\mathbb{T}} \to Set^{\mathbb{T}}$ being topological, Proposition 1.2 and Corollary 1.1 still hold for semi-Abelian algebras.

Because of the categorical method of the proofs, we have also the analogue of Propositions 1.3 and 1.4, of Theorem 1.1, Corollary 1.2 and Remark 1.1. This is the case also for Proposition 3.2 and the first part of Proposition 3.3. The analogue of Remark 1.2 is the following:

Remark 6.1. Given any topological semi-Abelian algebra A, any of its non-empty open or closed sets is necessarily an arbitrary union of equivalence classes of the congruence R_A^{Hf}.

It is straightforward to define the clopen topological semi-Abelian algebras, and we shall denote $Top_{co}^{\mathbb{T}}$, the full subcategory of $Top^{\mathbb{T}}$ determined by these objects. The analogue of Proposition 4.1 (characterization), Proposition 4.2 and Proposition 4.3 still hold. The only Hausdorff clopen topological semi-algebras are the discrete ones. The question is now: is there an analogue of Proposition 4.4. The aim of this last section is to show that the answer is positive.

Again, since in the semi-Abelian category $Set^{\mathbb{T}}$ normal subobjects of an algebra A are in bijection with congruences on A and since the characterization given by Proposition 4.1 still holds, the category $Top_{co}^{\mathbb{T}}$ is equivalent to the category $Rel(Set^{\mathbb{T}})$ of congruences in $Set^{\mathbb{T}}$ (given a congruence R on A, we shall denote by Θ_A^R the associated clopen topology on A which has the arbitrary unions of equivalence classes of R as open sets).

So let us begin, more generally, to investigate what are the normal subobjects in the category $Rel(\mathbb{C})$ when $\mathbb{C}$ is pointed protomodular. An object in $Rel(\mathbb{C})$ is a pair (Z, S) of an object Z of $\mathbb{C}$ and an internal equivalence relation S on Z. A morphism $f:(Z,S)\to(Z',S')$ is given by a map $f:Z\to Z'$ in $\mathbb{C}$ such that $S\subset f^{-1}(S')$. An equivalence relation $(R,W)\rightarrowtail(Z,S)\times(Z,S)$ in $Rel(\mathbb{C})$ is just a double equivalence relation in $\mathbb{C}$:

$$\begin{tikzcd} W \arrow[r, shift left, "d_0^R"] \arrow[r, shift right, "d_1^R"'] \arrow[d, shift right, "d_0^S"'] \arrow[d, shift left, "d_1^S"] & R \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ S \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] & Z \end{tikzcd}$$

The normal subobject associated with it is the following levelwise normal morphism in $Rel(\mathbb{C})$, where $x:X\rightarrowtail Z$ is normal to R and $l:L\rightarrowtail S$ is normal to W:

$$\begin{tikzcd} L \arrow[r, "l"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & S \arrow[d, shift right, "\delta_0"'] \arrow[d, shift left, "\delta_1"] \\ X \arrow[r, "x"'] & Z \end{tikzcd}$$

Conversely, thanks to the bijection between normal subobjects and internal equivalence relations, any levelwise normal morphism in $Rel(\mathbb{C})$ allows to restore a double equivalence relation and to produces a normal subobject in $Rel(\mathbb{C})$.

On the other hand, given any pair (R,S) of equivalence relations on Z, there is a larger double equivalence relation $R\square S$, called *the parallelistic double relation associated with* (R,S) and given by the inverse image of the equivalence relation $S\times S$ on $Z\times Z$ along the map $(d_0,d_1):R\rightarrowtail Z\times Z$. When $\mathbb{C}=Set$, we get $R\square S=\{(x,y,t,z)\mid xRy, tRz, xSt, ySz\}$, a situation we shall represent the following way:

$$\begin{tikzcd} x \arrow[r, dash, "S"] \arrow[d, dash, "R"'] & t \arrow[d, dash, "R"] \\ y \arrow[r, dash, "S"'] & z \end{tikzcd}$$

The normal subobject in $Rel(\mathbb{C})$ associated with the equivalence relation $(R, R\square S)$ on (Z,S) is the following map where the map x is normal to R:

$$x:\big(X,x^{-1}(S)\big)=\big(X,x^{-1}(R\cap S)\big)\rightarrowtail(Z,S).$$

Let us recall now the following result of [12]:

Theorem 6.1. *Let $\mathbb{C}$ be a protomodular category, R and S two equivalence relations on Z, and $x:X\rightarrowtail Z$ the normal subobject associated with R. Then R and S are connected* (*i.e.* $[R,S]=0$) *if and only if the map $X\xrightarrow{x}Z\xrightarrow{s_{0,S}}S$ is normal in $\mathbb{C}$, where $s_{0,S}$ denotes the subdiagonal determining the reflexivity of the equivalence relation S.*

We have then the following corollary:

Corollary 6.1. *Suppose $\mathbb{C}$ pointed protomodular, then $[R,S]=0$ if and only if the subobject $x:(X,\Delta_X)\rightarrowtail(Z,S)$ is normal in $Rel(\mathbb{C})$, where Δ_X is the discrete relation on X.*

Proof. Suppose $[R,S]=0$. Let $p: R\times_Z S\to Z$ $(aRbSc)\rightarrowtail p(a,b,c)$ be the connector between R and S, and $Ch[p]$ the Chasles double relation on Z associated with the connector p:

$$\begin{tikzcd} Ch[p] \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & S \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ R \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & Z \end{tikzcd}$$

Recall that the elements of $Ch[p]$ are those elements of $(x,y,t,z)\in R\square S$ which satisfy $t=p(x,y,z)$.

Then a levelwise argument shows that the map $x:(X,\Delta_X)\rightarrowtail(Z,S)$ is normal to $(R,Ch[p])$ in the category $Rel(\mathbb{C})$.

Conversely suppose $x:(X,\Delta_X)\rightarrowtail(Z,S)$ is normal in $Rel(\mathbb{C})$. This implies that $X\xrightarrow{x}Z\xrightarrow{s_{0,S}}S$ is normal in $\mathbb{C}$, and accordingly that $[R,S]=0$. □

We can now step forward:

Proposition 6.3. *Let $\mathbb{C}$ be a finitely cocomplete exact pointed protomodular category. Let R be an equivalence relation on Z, and $x:X\rightarrowtail Z$ be its associated normal subobject. Let S be an other equivalence relation on Z. Then any equivalence relation T on Z such that $[R,S]\subset T\subset R\cap S$ produces a normal subobject $x:(X,x^{-1}(T))\rightarrowtail(Z,S)$ in $Rel(\mathbb{C})$, above the map $x:X\rightarrowtail Z$ in $\mathbb{C}$.*

Proof. Consider $\rho: Z\twoheadrightarrow Z/T$ the quotient map. Then the direct image $\rho(x):\rho(X)\rightarrowtail Z/R$ of the subobject x by ρ is normal to the direct image $\rho(R)$. Since we have $[R,S]\subset T$, the equivalence relations $\rho(R)$ and $\rho(S)$ on Z/T are connected. Thus, according to the previous corollary, the map $\rho(x):(\rho(X),\Delta_{\rho(X)})\rightarrowtail(Z/T,\rho(S))$ is normal in $Rel(\mathbb{C})$. Now, since we have $T\subset R\cap S$, we have also $R=\rho^{-1}(\rho(R))$ and $S=\rho^{-1}(\rho(S))$. But, in any protomodular category, the normal subobjects are stable under inverse image, so that $x:(X,R[\rho.x])\rightarrowtail(Z,S)$ is normal in $Rel(\mathbb{C})$, where $R[\rho.x]$ denotes the kernel equivalence relation of the map $\rho.x$. But we have also: $R[\rho.x]=x^{-1}(R[\rho])=x^{-1}(T)$. □

In order to prove the converse, we need to introduce the following construction. Consider any double equivalence relation on Z in $Rel(\mathbb{C})$, with $\mathbb{C}$ finitely complete:

$$\begin{tikzcd} W \arrow[r, shift left, "d_0^R"] \arrow[r, shift right, "d_1^R"'] \arrow[d, shift right, "d_0^S"'] \arrow[d, shift left, "d_1^S"] & R \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ S \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] & Z \end{tikzcd}$$

When $\mathbb{C} = Set$, an object w in W will be represented by the following rectangle, with $(x, y, t, z) = d(w) \in Z^4$:

$$\begin{tikzcd} x \arrow[r, dash, "S"] \arrow[d, dash, "R"'] & t \arrow[d, dash, "R"] \\ y \arrow[r, dash, "S"'] \arrow[ur, phantom, "w"] & z \end{tikzcd}$$

Definition 6.1. We call *contraction* of the double equivalence relation W the relation $\overline{W}$ on Z given by the following pullback:

$$\begin{tikzcd} \overline{W} \arrow[r] \arrow[d] & W \arrow[d, "d"] \\ Z \times Z \arrow[r, "(Id, s_0 . p_1)"'] & (Z \times Z)^2 \end{tikzcd}$$

In *Set*, we have $x \overline{W} y$ if and only if:

$$\begin{tikzcd} x \arrow[r, dash, "S"] \arrow[d, dash, "R"'] & y \arrow[d, dash, "R"] \\ y \arrow[r, dash, "S"'] \arrow[ur, phantom, "w"] & y \end{tikzcd}$$

Example 6.1. When $W = R\square S$, then $\overline{W} = R \cap S$. When $[R, S] = 0$, then $\overline{Ch[p]} = \Delta_X$.

Proposition 6.4. *The contraction $\overline{W}$ is an equivalence relation on Z, such that $\overline{W} \subset R \cap S$. Moreover it determines a fibrant map in $Rel(\mathbb{C})$:*

$$\begin{tikzcd} R\square\overline{W} \arrow[r, "\bar{i}_S"] \arrow[d, shift right, "d_0^{\overline{W}}"'] \arrow[d, shift left, "d_1^{\overline{W}}"] & W \arrow[d, shift right, "d_0^S"'] \arrow[d, shift left, "d_1^S"] \\ \overline{W} \arrow[r, "i_S"'] & S \end{tikzcd}$$

Proof. The first assertion is straightforward. The second assertion means that any of the previous commutative squares is a pullback. Thanks to the Yoneda embedding, we only have to prove the assertion in the category *Set*. Consider an element $(x, y, t, z) \in R\square\overline{W}$. Then we have:

$$\begin{tikzcd} t \arrow[r, dash, "R"] \arrow[d, dash, "S"'] & x \arrow[r, dash, "R"] \arrow[d, dash, "S"] & y \arrow[r, dash, "R"] \arrow[d, dash, "S"'] & z \arrow[d, dash, "S"] \\ t \arrow[r, dash, "R"'] \arrow[ur, phantom, "w"] & t \arrow[r, dash, "R"'] & z \arrow[r, dash, "R"'] \arrow[ur, phantom, "w"] & z \end{tikzcd}$$

So we get the following situation, the middle square being given by the reflexivity of W above R:

$$\begin{tikzcd}
x \arrow[r, dash, "R"] \arrow[d, dash, "S"'] \arrow[dr, phantom, "w"] & t \arrow[r, dash, "R"] \arrow[d, dash] \arrow[dr, phantom, "w"] & z \arrow[r, dash, "R"] \arrow[d, dash] \arrow[dr, phantom, "w"] & y \arrow[d, dash, "S"] \\
t \arrow[r, dash, "R"'] & t \arrow[r, dash, "R"'] & z \arrow[r, dash, "R"'] & z
\end{tikzcd}$$

and we can then fulfil the following square which produces the map $\tilde{i}_S$:

$$\begin{tikzcd}
x \arrow[r, dash, "R"] \arrow[d, dash, "S"'] \arrow[dr, phantom, "w"] & y \arrow[d, dash, "S"] \\
t \arrow[r, dash, "R"'] & z
\end{tikzcd}$$

It remains to check that the diagrams in the statement are pullbacks by a routine calculation. □

Corollary 6.2. *Let $x : X \rightarrowtail Z$ be the normal subobject associated with R, then the map $x^{-1}(\overline{W}) \overset{\tilde{x}}{\rightarrowtail} \overline{W} \overset{j_S}{\rightarrowtail} S$ is normal to W.*

Proof. The inverse image $x^{-1}(\overline{W})$ is given by the following pullback:

$$\begin{tikzcd}
x^{-1}(\overline{W}) \arrow[r, "\tilde{x}"] \arrow[d, "{(d_0,d_1)}"'] & \overline{W} \arrow[d, "{(d_0,d_1)}"] \\
X \times X \arrow[r, "x \times x"'] & Z \times Z
\end{tikzcd}$$

The map x is normal to R. Accordingly $x \times x$ is normal to $R \times R$ and the map $\tilde{x}$ to the inverse image of $R \times R$ along $(d_0, d_1) : \overline{W} \rightarrowtail Z \times Z$, which is $R \square \overline{W}$. Consequently the left-hand side morphism of equivalence relations is fibrant:

$$\begin{tikzcd}
(x^{-1}(\overline{W}))^2 \arrow[r] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & R \square \overline{W} \arrow[r] \arrow[d, shift right, "d_0^{\overline{W}}"'] \arrow[d, shift left, "d_1^{\overline{W}}"] & W \arrow[d, shift right, "d_0^S"'] \arrow[d, shift left, "d_1^S"] \\
x^{-1}(\overline{W}) \arrow[r, "\tilde{x}"'] & \overline{W} \arrow[r, "j_S"'] & S
\end{tikzcd}$$

while, by the previous proposition, the right-hand side morphism is fibrant; so that the composite is fibrant, and the assertion is proved. □

Whence:

Theorem 6.2. *Let $\mathbb{C}$ be a pointed exact protomodular category. Let R be an equivalence relation on Z, and $x : X \rightarrowtail Z$ be its associated normal subobject. Let S be any other equivalence relation on Z. Then there are as many normal subobjects to (Z, S) in Rel$(\mathbb{C})$ above x as equivalence relations T on Z such that $[R, S] \subset T \subset R \cap S$.*

Proof. The direct part of this theorem is just Proposition 6.3. Conversely suppose $x:(X,D) \rightarrowtail (Z,R)$ is a normal subobject in $Rel(\mathbb{C})$. Let us denote by (R,W) the equivalence relation to which it is normal. Let us set $T = \overline{W}$. Then, by Corollary 6.2 and the bijection between normal subobjects and internal equivalence relations, we have $D \simeq x^{-1}(\overline{W}) = x^{-1}(T)$. It remains to show that $[R,S] \subset T = \overline{W}$. For that let us consider the map $\rho : Z \twoheadrightarrow Z/\overline{W}$ and the following diagram where $\rho(x)$ is the direct image of x (which is normal to $\rho(R)$), and $\rho(S)$ is the direct image of the equivalence relation S:

$$\begin{tikzcd}
x^{-1}(\overline{W}) \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, "\tilde{x}"'] & X \arrow[r, two heads, "\rho_X"] \arrow[d, "x"] & \rho(X) \arrow[d, "\rho(x)"] \\
\overline{W} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[dr, "j_S"'] & Z \arrow[r, two heads, "\rho"'] \arrow[d, dotted, "s_0"] & Z/\overline{W} \arrow[d, "s_0"] \\
 & S \arrow[r, two heads, "\rho_S"'] & \rho(S)
\end{tikzcd}$$

The map $\rho(x)$ being a monomorphism, the kernel equivalence relation of ρ_X is $x^{-1}(\overline{W})$. Let us set $\bar{\rho} = \rho_X.d_0 = \rho_X.d_1$. This is a regular epi. Now, the monomorphism $j_S.\tilde{x}$ is normal (to W), and we have $(s_0.\rho(x)).\bar{\rho} = \rho_S.(j_S.\tilde{x})$. This means that the direct image of this monomorphism along the map ρ_S is thus $s_0.\rho(x)$. This last map is normal itself, since it is the direct image of the normal monomorphism $j_S.\tilde{x}$ (see the previous corollary). Consequently $\rho(R)$ and $\rho(S)$ are connected by Theorem 6.1. By the universal property of $[R,S]$, we have then $[R,S] \subset R[\rho] = \overline{W}$.

It remains to check that the construction given here is the inverse of the one given in Proposition 6.3. This will be a straightforward consequence of the following observation. □

Proposition 6.5. *Let $\mathbb{C}$ be a protomodular category, and $x : X \rightarrowtail Z$ a normal monomorphism (to R). Then taking the inverse image along x of equivalence relations smaller than R is injective.*

Proof. Let D_1 and D_2 be two equivalence relations on Z such that $x^{-1}(D_1) \simeq x^{-1}(D_2)$, and $D_i \subset R$ for $i \in \{1,2\}$. Then the following diagram in $Rel(\mathbb{C})$ is a pullback since the two horizontal arrows are inverse images:

$$\begin{tikzcd}
(X, x^{-1}(D_i)) \arrow[r, "x"] \arrow[d] & (Z, D_i) \arrow[d] \\
(X, \nabla_X) \arrow[r, "x"'] & (Z, R)
\end{tikzcd}$$

But the lower map is fibrant (x normal to R), so that the upper map is fibrant too. Now $\mathbb{C}$ is protomodular, and the fibrant maps in $Rel(\mathbb{C})$ are co-Cartesian with respect to the forgetful functor $Rel(\mathbb{C}) \to \mathbb{C}$, see [12]. Accordingly $D_1 \simeq D_2$. □

Now, given any semi-Abelian theory $\mathbb{T}$, we can assert the result we were aiming for:

Theorem 6.3. *Given any clopen topological semi-Abelian algebra (Z, Θ_Z^S) and any normal subalgebra $x : X \rightarrowtail Z$ (to R), there are as many clopen topological normal subobjects of (Z, Θ_Z^S) above x as congruences T such that $[R, S] \subset T \subset R \cap S$. The associated normal subobject with this T is the map $x : (X, \Theta_X^{x^{-1}(T)}) \rightarrowtail (Z, \Theta_Z^S)$.*

Proof. Straightforward, considering first that the category $Top_{co}^{\mathbb{T}}$ is isomorphic to the category $Rel(Set^{\mathbb{T}})$, and that $Set^{\mathbb{T}}$ is pointed, exact and protomodular, and then applying Theorem 6.2. □

6.2. The normal subobjects in the category $Top^{\mathbb{T}}$

The characterization of the normal subobjects in $Top^{\mathbb{T}}$ is clearly dependent on the characterization of the normal subalgebras in the semi-Abelian category $Set^{\mathbb{T}}$, which is done by the Theorem 3.2.13 in [3] and is far from being easy. But, for the classical algebraic theories, with very simple characterization of normal subalgebras, this is fairly accessible. We shall briefly sketch here the case of topological (non-unitary) commutative rings, mimicking what we did for the topological groups. Of course a normal subalgebra in the category *CMRg* of commutative rings is an ideal.

Let $\mathbb{E}$ be any finitely complete category, and $CMRg(\mathbb{E})$ the category of internal commutative rings in $\mathbb{E}$. We shall present here an internal commutative ring as an object X of $\mathbb{E}$ endowed with a subtraction $\delta : X \times X \to X$ (rather than an addition) and a multiplication $\mu : X \times X \to X$. Let (I, δ_I, μ_I) be any subring of (X, δ, μ), and let us consider now the following pullback:

$$\begin{tikzcd} \Gamma_X^I \arrow[r, "\bar{\delta}"] \arrow[d, "j"'] & I \arrow[d, "i"] \\ X \times X \arrow[r, "\delta"'] & X \end{tikzcd}$$

This is clearly internally corresponding to $\Gamma_X^I = \{(u, v) \in X \times X \mid v - u \in I\}$.

Lemma 6.1. *The map $j : \Gamma_X^I \rightarrowtail X \times X$ defines, on the object X in $\mathbb{E}$, an internal equivalence relation which actually lies in the category $Ab(\mathbb{E})$ of internal Abelian groups in $\mathbb{C}$ and to which the subobject $i : I \rightarrowtail X$ is normal in $Ab(\mathbb{E})$.*

Of course, there is no reason why Γ_X^I would be a subring of the ring $X \times X$. Categorically speaking, this is the case if and only if, in the category $\mathbb{E}$, the left-hand side vertical map in the following pullback is an isomorphism, where the map $\mu_{X \times X}$ denotes the multiplication of the product ring:

$$\begin{tikzcd} P \arrow[rr] \arrow[d] & & \Gamma_X^I \arrow[d, "j"] \\ \Gamma_X^I \times \Gamma_X^I \arrow[r, "j \times j"'] & (X \times X)^2 \arrow[r, "\mu_{X \times X}"'] & X \times X \end{tikzcd}$$

Proposition 6.6. *The subobject $i : I \rightarrowtail X$ is normal in CMRg*$(\mathbb{E})$ *if and only if Γ_X^I is a subring of the product ring $X \times X$.*

There is also the following characterization, which is the internal translation, thanks to the Yoneda embedding, of a well-known result in the set theoretical context:

Proposition 6.7. *The map $i : I \rightarrowtail X$ is normal in CMRg*$(\mathbb{E})$ *if and only if, in the category $\mathbb{E}$ the left-hand side vertical map in the following pullback is an isomorphism*:

$$\begin{tikzcd}
J \arrow[rr] \arrow[d] & & I \arrow[d, "i"] \\
I \times X \arrow[r, "i \times X"'] & X \times X \arrow[r, "\mu"'] & X
\end{tikzcd}$$

Proof. This is the translation of the fact that, for all $(y, x) \in I \times X$, the element $y.x$ must be in I. $\square$

Let us pass to the category *CMRg*(Top). Let (X, T_X) a topological ring, and (I, T_I) a subobject in *CMRg*(Top). Then T_I is such that the inclusion $i : I \rightarrowtail X$ of rings is continuous, but, again, T_I is not necessarily the topology T_I^i induced by T_X on I. According to the previous characterization, a subobject $i : (I, T_I) \rightarrowtail (X, T_X)$ is normal in *CMRg*(Top) if and only if:

(1) the subring I is an ideal of X,
(2) the map $\mu_* : I \times X \to I$; $(y, x) \mapsto y.x$ is continuous from the topological product $(I, T_I) \times (X, T_X)$ to (I, T_I).

References

[1] J. Adamek, H. Herrlich, G.E. Strecker, Abstract and Concrete Categories, Wiley, New York, 1990.
[2] M. Barr, Exact Categories, Lecture Notes in Math., vol. 236, Springer, Berlin, 1971, pp. 1–120.
[3] F. Borceux, D. Bourn, Mal'cev, Protomodular, Homological and Semi-Abelian Categories, Math. Appl., vol. 566, Kluwer Academic, Dordrecht, 2004, 479 p.
[4] F. Borceux, M.M. Clementino, Topological semi-Abelian algebras, Adv. Math. 190 (2005) 425–453.
[5] N. Bourbaki, Eléments de Mathématiques, Groupes topologiques, chap. 3, Hermann, Paris, 1951.
[6] D. Bourn, The shift functor and the comprehensive factorization for internal groupoids, Cahiers Topologie Géom. Différentielle Catégoriques 28 (1987) 197–226.
[7] D. Bourn, Normalization equivalence, kernel equivalence, and affine categories, in: Lecture Notes in Math., vol. 1488, Springer, Berlin, 1991, pp. 43–62.
[8] D. Bourn, Normal subobjects and Abelian objects in protomodular categories, J. Algebra 228 (2000) 143–164.
[9] D. Bourn, 3×3 lemma and protomodularity, J. Algebra 236 (2001) 778–795.
[10] D. Bourn, Commutator theory in regular Mal'cev categories, in: G. Janelidze, B. Pareigis, W. Tholen (Eds.), Galois Theory, Hopf Algebras, and Semi-Abelian Categories, in: Fields Institute Communications, vol. 43, American Mathematical Society, Providence, RI, 2004, pp. 61–75.
[11] D. Bourn, Commutator theory in strongly semi-Abelian categories, Theory Appl. Categories 13 (2004) 27–40.

[12] D. Bourn, M. Gran, Centrality and normality in protomodular categories, Theory Appl. Categories 9 (2002) 151–165.

[13] D. Bourn, G. Janelidze, Characterization of protomodular varieties of Universal Algebra, Theory Appl. Categories 11 (2003) 143–147.

[14] P.J. Higgins, An Introduction to Topological Groups, London Math. Soc. Lecture Notes Ser., vol. 15, Cambridge University Press, Cambridge, 1974.

[15] P.J. Higgins, Groups with multiple operators, Proc. London Math. Soc. (1956) 366–416.

[16] G. Janelidze, L. Marki, W. Tholen, Semi-Abelian categories, J. Pure Appl. Algebra 168 (2002) 367–386.

[17] O. Wyler, Taut lift theorem, Arch. Math. 22 (1971) 7–17.